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On the arithmetical rank of squarefree monomial ideals concerned with the complete bipartite graph $K_{2,n}$

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Introduction

S: a polynomial ring over a field K.
$I, J \subset S$: squarefree monomial ideals.

Let $X = V(I)$ be an affine algebraic set. Then how many hypersurfaces need to express X as intersection of those?

$$X = X_0 \cap X_1 \cap \cdots \cap X_{s-1}, \quad X_i = V(f_i).$$

The **arithmetical rank** of I:

$$\operatorname{ara} I := \min\{s \;:\; \exists f_0, f_1, \ldots, f_{s-1} \in I \text{ s. t. } \sqrt{(f_0, f_1, \ldots, f_{s-1})} = \sqrt{I}\}.$$

Fact (Lyubeznik).

$$(\star) \qquad \operatorname{height} I \le \operatorname{pd}_S S/I \le \operatorname{ara} I \le \mu(I).$$

Problem 0.1. *When does* $\operatorname{ara} I = \operatorname{pd}_S S/I$ *hold?*

<u>**Known results**</u>:
(1) $\mu(I) - \operatorname{height} I = 0$ (clear).
(2) (KTY) $\mu(I) - \operatorname{height} I = 1$.
(3) (KTY) $\mu(I) - \operatorname{height} I = 2$.
(4) $\mu(I) - \operatorname{pd}_S S/I = 0$ (clear).
(5) (KTY) $\mu(I) - \operatorname{pd}_S S/I = 1$.
(1)* (Schenzel–Vogel, Schmitt–Vogel) $\operatorname{arithdeg} I - \operatorname{indeg} I = 0$.
(2)* (KTY) $\operatorname{arithdeg} I - \operatorname{indeg} I = 1$.
(4)* (KTY) $\operatorname{arithdeg} I - \operatorname{reg} I = 0$.

<u>**Counterexamples**</u> (char $K \neq 2$):
(6) (Yan) Stanley–Reisner ideal associated to Reisner's triangulation of the projective plane.
(7) (KTY) An ideal with $\mu(I) - \operatorname{height} I = 3$.
(7)* (KTY) An ideal with $\operatorname{arithdeg} I - \operatorname{indeg} I = 3$.

($(\cdot)^*$ is Alexander dual of $(\cdot)$.)
In this poster, we focus on the case corresponding to **(3)*** and **(5)***.

1 Alexander duality

Example 1.1 (Alexander dual ideal).

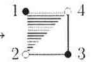
$$(\clubsuit) \quad I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_1, x_4) \cap (x_2, x_5) \cap (x_3, x_6)$$
$$\Longrightarrow I^* = (x_1x_2x_3, x_4x_5x_6, x_1x_4, x_2x_5, x_3x_6).$$

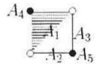
<u>**Properties**</u>:
- $I^{**} = I$.
- (Frühbis-Krüger–Terai)
  $\operatorname{indeg} I^* = \operatorname{height} I. \quad \operatorname{reg} I^* = \operatorname{pd}_S S/I. \quad \operatorname{arithdeg} I^* = \mu(I).$
- (Hoa–Trung, Frühbis-Krüger–Terai)
  $(\star)^* \qquad \operatorname{indeg} I \le \operatorname{reg} I \le \operatorname{arithdeg} I.$

2 Hypergraphs

Example 2.1 (Hypergraph).

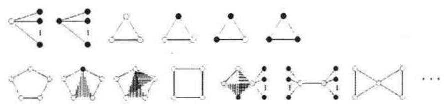
$$J = (x_1x_4, x_1x_2, x_2x_3x_5, x_1x_3) \longrightarrow$$





$$\longrightarrow J = (A_1A_4, A_1A_2, A_2A_3A_5, A_1A_3)$$

Using hypergraph, we can classify squarefree monomial ideals with $\mu(I) - \operatorname{height} I \le 2$ (KTY).



3 The case of $(3)^*$ $\operatorname{arithdeg} I - \operatorname{indeg} I = 2$

There are 3 cases by $(\star)^*$:
(a) $\operatorname{arithdeg} I = \operatorname{reg} I = \operatorname{indeg} I + 2$.
(b) $\operatorname{arithdeg} I = \operatorname{reg} I + 1 = \operatorname{indeg} I + 2$.
(c) $\operatorname{arithdeg} I = \operatorname{reg} I + 2 = \operatorname{indeg} I + 2$.

The case (a): contained in $(4)^*$.

Theorem 3.1 (KTY). *For the ideal I of* **the case (c)**,

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

Proof. We determine the arithmetical rank according to the classification by hypergraphs. □

The case (b) is an open problem. This case is a part of the case $(5)^*$:
$(5)^*$ $\operatorname{arithdeg} I - \operatorname{reg} I = 1$.
$\Longleftrightarrow \mathcal{H}(I^*)$ **"contains" a complete bipartite graph** (Terai).

Problem 3.2.

$$\mathcal{H}(J) = K_{2,n} =$$

Determine the arithmetical rank of $I = J^*$.

Case1: assign 1 variable to each edges.

Theorem 3.3 (KTY). *For any n, the ideal I which is obtained by assigning 1 variable to each edge of hypergraphs in Problem 3.2 satisfies*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

We shall see the case $n = 3$.

$$(\clubsuit) \quad I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3) \cap (x_1, y_1) \cap (x_2, y_2) \cap (x_3, y_3).$$

In this case, $\operatorname{pd}_S S/I = 3 = \operatorname{ara} I$:

$$\begin{cases} f_0 = x_1x_2x_3 \cdot y_1y_2y_3, \\ f_1 = x_1x_2 \cdot y_3 + x_1x_3 \cdot y_2 + x_2x_3 \cdot y_1, \\ f_2 = x_1 \cdot y_2y_3 + x_2 \cdot y_1y_3 + x_3 \cdot y_1y_2. \end{cases}$$

Case2: general cases.

Theorem 3.4 (KTY). *For $n = 2, 3, 4$, the ideal I in Problem 3.2 satisfies*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

The idea of the proof:

Example 3.5 (Known result $(4)^*$).

$$I = (y_1, x_1, x_2) \cap (y_2, x_1, x_3) \cap (y_3, x_2).$$

Then $\operatorname{ara} I = \operatorname{pd}_S S/I = 4$:

$$\begin{cases} h_0 = y_1y_2y_3, \\ h_1 = \mathbf{x_1} \cdot y_3 + \mathbf{x_2} \cdot y_2 + \mathbf{x_3} \cdot y_1y_3, \\ h_2 = \mathbf{x_1x_2} + \mathbf{x_1x_3} \cdot y_3 + \mathbf{x_2x_3}, \\ h_3 = \mathbf{x_1x_2x_3}. \end{cases}$$

Note that h_1, h_2, h_3 are **modifications of elementary symmetric functions** of x_1, x_2, x_3.

<u>**The case $n = 3$ of Theorem 3.4**</u>:

$$(\clubsuit)' \qquad I = P_1 \cap P_2 \cap Q_1 \cap Q_2 \cap Q_3,$$

where

$$P_1 = (\underline{X_1}, \underline{X_2}, \underline{X_3}), \quad P_2 = (\underline{Y_1}, \underline{Y_2}, \underline{Y_3}), \quad Q_i = (\underline{X_i}, \underline{Y_i}),$$
$$\underline{X_i} = x_i, \underline{X_i'} = x_i, x_{i2}, \ldots, x_{i\ell_i}$$
$$\underline{Y_i} = y_i, \underline{Y_i'} = y_i, y_{i2}, \ldots, y_{im_i}, \qquad (i = 1, 2, 3).$$

Then $\operatorname{pd}_S S/I = \sum_{i=1}^{3}(\ell_i + m_i) - 3$.

$$\begin{cases} k\textbf{th elementary symmetric functions} \\ \quad g_k' = S_k(\underline{X_1'} \cup \underline{X_2'} \cup \underline{X_3'} \cup \underline{Y_1'} \cup \underline{Y_2'} \cup \underline{Y_3'}) \\ \qquad\qquad (k = 1, 2, \ldots, \operatorname{pd}_S S/I - 3), \\ f_0, f_1, f_2, \qquad \operatorname{pd}_S S/I \textbf{ elements!} \text{ and modify them.} \end{cases}$$